November 4, hours 22-23: Compactness in R^n and in Metric Spaces

November-03-10
2:49 PM

(week 8)

No class Monday!

Read Along. Still Munkres 26, 27 maybe 48.

Goal. Compactness in $\mathbb{R}^n$ & in metric spaces.

Thm. The image of a compact set by a continuous function is compact. **Cor:** The max value Thm.

Thm. A finite product of compact spaces is compact (& vice-versa, if all spaces are non-empty)

Corollary. A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

The Lebesgue Number Lemma. Given a cover $\mathcal{U}=\{U_\alpha\}$ of a compact metric space X, there exist $\delta_0>0$ s.t. $\forall x\in X\ \exists\alpha\ B(x,\delta_0)\subset U_\alpha$.

Proof. Set $\Delta(x)=\sup\{\delta\leq 1:\exists\alpha\ B(x,\delta)\subset U_\alpha\}$
if $d(x,y)<\epsilon$, then $\Delta(y)\geq\Delta(x)-\epsilon$. $[\text{so } |\Delta(x)-\Delta(y)|<\epsilon]$
take $\delta_0=\min\Delta$.

The Uniform Continuity Theorem. Def. A "uniformly cont. $f:X\to Y$, X,Y metric"

<u>Thm</u> X compact metric, Y metric, $f:X\to Y$ cont. $\Rightarrow$ f is uniformly cont.

The FIP. X is compact iff every collection of closed sets that has the fip has an non-trivial intersection.

Pf. (compact) $\Leftrightarrow$ (every collection of closed sets with empty intersection has a finite sub...) $\Leftrightarrow$ (The above.)

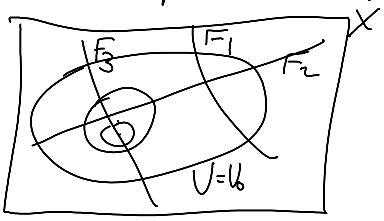
done line

Def. A Baire space is a space X s.t. any countable union of closed sets w/ empty interior has empty interior.

(in complements! Every countable intersection of open dense sets is dense) (counterexample: $\mathbb{Q}$)

Thm. A compact T_2 space is Baire.

Pf (Show that $U \not\subset \operatorname{int} \bigcup F_k$)



start with $U_0 = U$, construct U_n open s.t. $\overline{U_n} \subset U_{n-1} \setminus F_n \subset U \setminus \bigcup_{k \le n} F_k$

Thm. A compact T_2 space with no isolated points is uncountable.